

Quadratic maps between groups

Dedicated to Mamuka Jibladze on the occasion of his fiftieth birthday

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Abstract

The notion of quadratic maps between arbitrary groups appeared at several places in the literature on quadratic algebra. Here a unified extensive treatment of their properties is given; the relation with a relative version of Passi's polynomial maps and groups of degree 2 is established and used to study the structure of the latter.

Introduction. Polynomial maps appear in nilpotent group theory for a long time, originally in the form of rational (numerical) functions, for example in the Hall-Petrescu formula or the group law of torsionfree nilpotent groups when written with respect to a Mal'cev basis. An *intrinsic* notion of polynomial maps from groups to *abelian* groups was introduced by Passi [30], together with a universal example $G \rightarrow P_n(G)$ where the abelian group $P_n(G)$ is called “polynomial group”. Passi's motivation came from the study of dimension subgroups; since then, his construction turned out to provide a key tool in the study of many other problems: in the theory of group schemes [7] as well as in the theory of nilpotent groups, concerning their second (co)homology [15], [16], automorphism groups or simplicial objects [13], [17]. However, a need to study polynomial maps between *arbitrary* groups comes from unstable homotopy theory; after Baues' [3] and the author's [12] study of metastable homotopy groups and Moore spaces [4] the foundations of “quadratic algebra” were laid in [5] where a notion of quadratic maps with non-abelian target group first appeared. Since then, in the steadily growing literature on quadratic algebra and its applications, various variants and properties were exhibited when needed, in work of Baues, Jibladze, Muro, Pirashvili and the author

(most of these articles can be found on ArXiv). So the purpose of this paper is to provide a thorough unified treatment of what is called *weakly* quadratic maps, because of their good functorial behaviour; for brevity we drop the word “weakly” in this paper. So several of our formulas and properties appear also elsewhere in the literature, but we include and prove most of them here for the sake of a coherent exposition. However, we here work in the slightly more general framework of quadratic maps *relative to a subgroup*, inspired by Passi’s study of relative dimension subgroups, see [32], lateron extended by Kuz’min [24] and in [20]. The relative viewpoint here leads to the construction of various categories of relative quadratic maps generalizing the “quadratic envelope of the category of 2-step nilpotent groups” introduced by Jibladze and Pirashvili [23]. In particular, the category **CP** of central quadratic pair maps introduced here turns out to play a fundamental role in quadratic algebra since it allows to refound the basic notions and to define modules over square ringoids instead of only square groups as in [5]; this is work in progress and will be presented in [11] and in a forthcoming book on quadratic algebra jointly written with H. Gaudier, F. Goichot, B. Loiseau and T. Pirashvili.

In section 1 we introduce and study relative quadratic maps and their universal examples $G \rightarrow Q(G, B)$ which constitute a nonabelian version of the relative version of Passi’s polynomial maps and groups in degree 2. The latter are studied in section 2, first for arbitrary degree before focussing on the degree 2 where the relative Passi groups $P_2(G, B)$ turn out to be precisely the abelianization of the groups $Q(G, B)$; this fact allows to deduce their structure from the (easy) non-abelian case in a rather simple way, and to deduce several exact sequences for $P_2(G, B)$.

Finally we note that most of the theory of this paper can be generalized to the abstract setting of semi-abelian categories [21]; in particular, all algebraic theories containing a group law as part of the structure (for example, algebras over any non-unitary k -linear operad, like Lie algebras), admit a theory of quadratic maps. Also, a theory of quadratic maps between modules is inaugurated in [10], and a notion of polynomial maps between non-abelian groups of arbitrary degree is introduced by the author, from an inductive viewpoint generalizing the one adopted in this paper; this is work in progress. There is, however, a different approach due to Leibman which also has interesting applications [26], [27]; the precise relation between the two approaches remains to be clarified.

1 Quadratic maps between groups

Throughout this paper, the symbols G and H denote groups. The commutator of elements $a, b \in G$ is defined as $[a, b] = aba^{-1}b^{-1}$, and the conjugation is denoted by ${}^ab = aba^{-1}$. We write $G^{ab} = G/G'$ and $ab : G \twoheadrightarrow G^{ab}$ for the natural projection. Recall that the lower central series of G is defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) =$

$[G, \gamma_i(G)]$. We say that G is n -step nilpotent if $\gamma_{n+1}(G) = \{1\}$. If G is 2-step nilpotent, the commutator map $[-, -] : G \times G \rightarrow G$ is wellknown to be bilinear.

Let $f : G \rightarrow H$ be some function between *arbitrary* groups. We shall, however, write the group law of H *additively* since in many applications H is abelian, and in those in [5], [11] where H is genuinely nonabelian, it is written additively anyway to match the conventions in homotopy theory which originally motivated these developments.

Define the *deviation function* or *cross effect* of f to be the map

$$d_f : G \times G \rightarrow H \quad \text{by} \quad d_f(a, b) = f(ab) - f(b) - f(a). \quad (1)$$

Furthermore, let I_f resp. D_f denote the subgroup of H generated by $\text{Im}(f)$ resp. $\text{Im}(d_f)$.

Definitions 1.1 We say that f as above is

- (a) *linear* if $d_f = 0$, i.e., f is a group homomorphism;
- (b) *quadratic* if d_f is *bilinear* and D_f is *central* in I_f , or more explicitly, $\forall a, b, c \in G, [d_f(a, b), f(c)] = 0$.

This definition of quadratic maps first appears in [23] under the same of weakly quadratic maps.

Note that linear maps are quadratic. We denote by $\text{Quad}(G, H)$ the set of quadratic maps from G to H .

We also need a relative version of quadratic maps, as follows. Let B be a subgroup of G . Then we say that f as above is *quadratic relative B* if f is quadratic and $d_f(B \times G) = d_f(G \times B) = 0$. Note that f is quadratic iff it is quadratic relative $\{1\}$. We define the *radical* of a quadratic map f to be the set $\text{rad}(f)$ consisting of all elements a of G such that for all $b \in G$, $d_f(a, b) = d_f(b, a) = 0$. Note that $\text{rad}(f)$ is a subgroup of G and that $G' \subset \text{rad}(f)$ since d_f is a bilinear map taking values in the abelian group D_f . In particular, $\text{rad}(f)$ is normal and $G/\text{rad}(f)$ is abelian. It is also clear that $\text{rad}(f)$ is the largest subgroup B of G such that f is quadratic relative B .

In the following proposition we collect the basic properties of quadratic maps which are easily deduced from the definition.

Proposition 1.2 *Let $f : G \rightarrow H$ be a quadratic map relative some subgroup B of G .*

- (a) *One has the following identities for $a, b \in G$:*

$$f(ab) = d_f(a, b) + f(a) + f(b) = f(a) + f(b) + d_f(a, b) \quad (2)$$

$$d_f(a, b) = -f(a) + f(ab) - f(b); \quad (3)$$

- (b) f is normalized, i.e., $f(1) = 0$;
- (c) the restriction of f to B is linear, whence $f(B)$ is a subgroup of H ;
- (d) there is a canonical linear map

$$w_f : G/BG' \otimes G/BG' \rightarrow H \quad (4)$$

such that $w_f(\bar{a} \otimes \bar{b}) = d_f(a, b)$ for $a, b \in G$. \square

Examples 1.3 (0) Let R be any ring and $a, b \in R$. Then the function $f : R \rightarrow R$, $f(x) = ax^2 + bx$, is a quadratic map between additive groups. More generally, quadratic forms on vector spaces are quadratic maps.

(1) For any subgroup B of G , the 2-fold diagonal map $\delta^2 : G \rightarrow G/BG' \otimes G/BG'$, $\delta(a) = \bar{a} \otimes \bar{a}$, is quadratic relative B .

(2) Let L be a 2-step nilpotent Lie algebra (i.e., $[[L, L], L] = 0$) over $\mathbb{Z}[\frac{1}{2}]$. Then a multiplicative group law on L is defined by the truncated Campbell-Baker-Hausdorff-formula, i.e., $x \circ y = x + y + \frac{1}{2}[x, y]$ for $x, y \in L$. In fact, (L, \circ) is a uniquely 2-divisible 2-step nilpotent group, and this construction provides a functorial equivalence between groups of this type and 2-step nilpotent Lie algebras over $\mathbb{Z}[\frac{1}{2}]$. This is a special case of a more general result of Lazard [25], providing “abelian models” for sufficiently divisible nilpotent groups. Now in the 2-step nilpotent case above, the identity map $id : (L, \circ) \rightarrow (L, +)$ is *quadratic* with $d_{id}(x, y) = \frac{1}{2}[x, y]$.

We note that generalizing the above equivalence of Lazard we constructed functorial abelian models for *arbitrary* 2-step nilpotent groups (actually, for central group extensions with abelian cokernel), cf. [13] and also [17]; this construction is based on the properties of relative quadratic maps with values in abelian groups, see section 2 below.

(3) Let k be some commutative ring with unit and G be the subgroup $1 + Tk[[T]]$ of the group of units of the algebra $k[[T]]$ of power series over k . For $n \geq 0$, let $c_n : G \rightarrow k$, $c_n(\sum_{i \geq 0} a_i T^i) = a_n$. Then c_1 is linear and c_2 is quadratic since $c_2(fg) = c_2(f) + c_2(g) + c_1(f)c_1(g)$, $f, g \in G$.

(4) Let $\underline{\Sigma}(n, 3n-3)$ denote the pointed homotopy category of suspensions ΣX of topological spaces X such that ΣX is an $(n-1)$ -connected $(3n-3)$ -dimensional CW-space. Then for $\Sigma X, \Sigma Y \in \underline{\Sigma}(n, 3n-3)$ the second James-Hopf-invariant

$$\gamma_2 : [\Sigma X, \Sigma Y] \longrightarrow [\Sigma X, \Sigma Y \wedge Y]$$

(see [37]) is a quadratic map where the group structure on $[\Sigma X, Z]$ is induced by the cogroup structure of ΣX . Moreover, if also $\Sigma Z \in \underline{\Sigma}(n, 3n-3)$ and $f : \Sigma X \rightarrow \Sigma Y$ is a continuous map then the map

$$[f]^* : [\Sigma Y, \Sigma Z] \longrightarrow [\Sigma X, \Sigma Z]$$

is a quadratic map, see [3, Appendix]. This example gave rise to the notion of quadratic categories, see [5].

More examples appear in the following proposition showing that quadratic maps are intimately related to 2-step nilpotent groups.

Proposition 1.4 *Let G be a group and $n \geq 2$. Then the following properties are equivalent.*

(1) G is 2-step nilpotent.

(2) The $(n-1)$ -fold multiplication map $\mu_{n-1} : G^n \rightarrow G$, $\mu_{n-1}(a_1, \dots, a_n) = a_1 \cdots a_n$, is a quadratic map.

(3) For all groups K and all linear maps $f_1, \dots, f_n : K \rightarrow G$, the product map $f_1 \cdots f_n : K \rightarrow G$, $x \mapsto f_1(x) \cdots f_n(x)$, is quadratic.

(4) The map $2_G : G \rightarrow G$, $a \mapsto a^2$, is quadratic.

Each of these implies

(5) The map $n_G : G \rightarrow G$, $a \mapsto a^n$, is quadratic.

Note that property (2) neatly generalizes the often useful fact that G is abelian iff μ_1 is linear.

Proof: We first note that if G is 2-step nilpotent the following relations hold for $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ in G^n :

$$a_1 b_1 \cdots a_n b_n = \prod_{1 \leq i < j \leq n} [b_i, a_j] (a_1 \cdots a_n) (b_1 \cdots b_n), \quad (5)$$

$$d_{\mu_{n-1}}(a, b) = \prod_{1 \leq i < j \leq n} [b_i, a_j]. \quad (6)$$

In fact, if $i < j$, we have $b_i a_j = [b_i, a_j] a_j b_i$; in shuffling all the factors b_i in the product $a_1 b_1 \cdots a_n b_n$ to the right, one introduces all the commutators $[b_i, a_j]$ with $i < j$. But these are central in G so can be gathered on the left which proves the first formula. It implies the second one as $d_{\mu_{n-1}}(a, b) = (a_1 b_1 \cdots a_n b_n) (b_1 \cdots b_n)^{-1} (a_1 \cdots a_n)^{-1}$.

Now we prove the desired equivalences.

(1) \Rightarrow (2). We have $D_{\mu_{n-1}} \subset G' \subset Z(G)$ by equation (6) and as G is 2-step nilpotent. Each commutator $[b_i, a_j]$ being bilinear identity (6) shows that $d_{\mu_{n-1}}$ is bilinear, too.

(2) \Rightarrow (3). The map $K \rightarrow G^n$, $x \mapsto (f_1(x), \dots, f_n(x))$, is linear. Hence by Proposition 1.8 below the composite map with the quadratic map $\mu_{n-1} : G^n \rightarrow G$ is still quadratic.

(3) \Rightarrow (4). It suffices to take two of the maps f_i equal to the identity of G and the others equal to the trivial map.

(4) \Rightarrow (1). Using (3) we get $d_{2_G}(a, b) = [-a, b]$, hence $[-a, b] \cdot^b [-a, b'] = [-a, bb'] = d_{2_G}(a, bb') = d_{2_G}(a, b)d_{2_G}(a, b') = [-a, b][-a, b']$, for $a, b, b' \in G$. It follows that ${}^b[-a, b'] = [-a, b']$, whence G' is central in G and G is 2-step nilpotent.

(5) It suffices to take $K = G$ and each $f_i = id$ in assertion (3). \square

We now recall some relations already proved in Lemma 2 in [23].

Proposition 1.5 *Let $f : G \rightarrow H$ be a quadratic map. Then the following relations hold for $a, b \in G$.*

$$f(a^{-1}) = -f(a) + d_f(a, a) \quad (7)$$

$$f(ab^{-1}) = f(a) - f(b) - d_f(ab^{-1}, b) \quad (8)$$

$$f[a, b] = [f(a), f(b)] + d_f(a, b) - d_f(b, a) \quad (9)$$

$$f({}^a b) = {}^{f(a)} f(b) + d_f(a, b) - d_f(b, a) \quad (10)$$

$$d_{-f}(a, b) = -d_f(b, a) - f[a, b] \quad (11)$$

We point out that relation (9) has a conceptual meaning which makes it a crucial ingredient in the structure theory in section 2; it also is the only relation which generalizes to the setting of semi-abelian categories where it again plays a key role [21].

Composition of quadratic maps.

The content of this section is essentially due to my former student O. Perriquet [33]. Composing two quadratic maps does not give a quadratic map in general, see Example 1.3(0). However, the following sufficient condition assuring stability under composition is often satisfied in practice, see [5], [11].

Definition 1.6 Let $K \xrightarrow{g} G \xrightarrow{f} H$ be two quadratic maps relative some subgroup A of K and B of G , resp. We say that the couple (f, g) is a *quadratic pair* if $g(A) \subset B$ and $d_f(D_g \times G) = d_f(G \times D_g) = 0$.

For example, if g and f are quadratic maps such that $D_g \subset G'$ then for $A = K'$ and $B = G'$, (f, g) is a quadratic pair by (9).

Proposition 1.7 *If (f, g) as above is a quadratic pair the composite map $f \circ g$ is quadratic relative A , with*

$$d_{f \circ g} = f_* d_g + (g \times g)^* d_f. \quad (12)$$

Moreover, if $D_g \subset BG'$, g induces a linear map $\bar{g} : K/AK' \rightarrow G/BG'$ and we have

$$w_{f \circ g} = f_* w_g + (\bar{g} \otimes \bar{g})^* w_f. \quad (13)$$

We call (12) or (13) the *derivation property* of a quadratic pair.

Proof: Writing G and H additively we have for $a, b \in K$

$$\begin{aligned} d_{f \circ g}(a, b) &= fg(ab) - fg(b) - fg(a) \\ &= f(d_g(a, b) + g(a) + g(b)) - fg(b) - fg(a) \\ &= f(d_g(a, b)) + f(g(a) + g(b)) - fg(b) - fg(a) \quad \text{since } d_f(D_g \times G) = 0 \\ &= f(d_g(a, b)) + d_f(g(a), g(b)) + fg(a) + fg(b) - fg(b) - fg(a) \\ &= f(d_g(a, b)) + d_f(g(a), g(b)). \end{aligned}$$

Next we check that $d_{f \circ g}$ is linear in the first variable; the argument for the second variable is similar. For $a' \in K$, we have

$$\begin{aligned} d_{f \circ g}(aa', b) &= f(d_g(aa', b)) + d_f(g(aa'), g(b)) \\ &= f(d_g(a, b) + d_g(a', b)) + d_f(g(a) + g(a') + d_g(a, a'), g(b)) \\ &\stackrel{(*)}{=} f(d_g(a, b)) + f(d_g(a', b)) + d_f(g(a), g(b)) + d_f(g(a'), g(b)) \\ &\stackrel{(**)}{=} f(d_g(a, b)) + d_f(g(a), g(b)) + f(d_g(a', b)) + d_f(g(a'), g(b)) \\ &= d_{f \circ g}(a, b) + d_{f \circ g}(a', b). \end{aligned}$$

Equations $(*)$ and $(**)$ hold since $d_f(D_g \times G) = 0$ and since $D_f \subset Z(I_f)$, resp.

Moreover, if a or b is in A then $d_{f \circ g}(a, b) = 0$ since then $d_g(a, b) = 0$ and $d_f(g(a), g(b)) = 0$ as $g(A) \subset B$. It remains to check that $D_{f \circ g} \subset Z(I_{f \circ g})$. Let $a, b, c \in K$. Then $[d_f(g(a), g(b)), f(g(c))] = 0$ since $D_f \subset Z(I_f)$, and

$$[f(d_g(a, b)), f(g(c))] = f[d_g(a, b), g(c)] - d_f(d_g(a, b), g(c)) + d_f(g(c), d_g(a, b))$$

by (9). But the first of the latter three terms is trivial since $D_g \subset Z(I_g)$, the other two since $d_f(D_g \times G) = d_f(G \times D_g) = 0$. \square

Corollary 1.8 *Pre- or postcomposing a quadratic map by a linear map gives a quadratic map. More precisely, if $f : G \rightarrow H$ is quadratic and $g : K \rightarrow G$, $h : H \rightarrow L$ are linear maps of groups then hfg is quadratic with $d_{hfg} = h_*(g \times g)^* d_f$.* \square

Corollary 1.9 *There is a functor $\text{Quad}(G, -)$ from the category of groups to the category of sets sending a group H to $\text{Quad}(G, H)$ and a homomorphism $f : H \rightarrow K$ to the map $f_* : \text{Quad}(G, H) \rightarrow \text{Quad}(G, K)$.*

The following relations are crucial in dealing with quadratic categories, see [5].

Proposition 1.10 *Let $f, g : G \rightarrow H$ be two functions between groups.*

- (1) *If $(2_H, f)$ is a quadratic pair then D_f is central in H .*
- (2) *If $(2_H, g)$ is a quadratic pair then the following relations holds for $a, b \in G$.*

$$\begin{aligned} d_{f+g}(a, b) &= d_f(a, b) + d_g(a, b) + d_{2_H}(g(a), f(b)) \\ &= d_f(a, b) + d_g(a, b) + [f(b), g(a)]. \end{aligned}$$

Proof: If 2_H is quadratic then $d_{2_H}(a, b) = [-a, b]$ by (3), whence (1) follows from the equations $[H, D_f] = [-H, D_f] = d_{2_H}(H \times D_f) = 0$. To prove (2) calculate

$$\begin{aligned} d_{f+g}(a, b) &= (f+g)(a+b) - (f+g)(b) - (f+g)(a) \\ &= f(a+b) + g(a+b) - g(b) - f(b) - g(a) - f(a) \\ &= d_f(a, b) + f(a) + f(b) + d_g(a, b) + g(a) + g(b) - g(b) - f(b) - g(a) - f(a) \\ &\stackrel{(*)}{=} d_f(a, b) + d_g(a, b) + f(a) + f(b) + g(a) - g(a) - f(b) + [f(b), g(a)] - f(a) \\ &= d_f(a, b) + d_g(a, b) + [f(b), g(a)]. \end{aligned}$$

The last equation follows from the fact that H' is central in H as H is 2-step nilpotent since 2_H is quadratic, see Proposition 1.4. Equation (*) is due to the fact that D_g is central in H by assertion (1). Finally, as H is 2-step nilpotent, $[f(b), g(a)] = -[g(a), f(b)] = [-g(a), f(b)] = d_{2_H}(g(a), f(b))$. \square

The category of quadratic pairs.

Definition 1.11 *A pair of groups (G, B) consists of a group G together with a subgroup B of G . A pair map $f : (G, B) \rightarrow (H, C)$ between pairs of groups is a function f from G to H such that $f(B) \subset C$. Moreover, a pair map f is a linear pair map if the function f is linear, and f is a quadratic pair map if f is quadratic relative B such that C contains D_f .*

Note that any quadratic map $f : G \rightarrow H$ is a quadratic pair map from (G, G') to $(H, H'D_f)$ by (9) and from $(G, \text{rad}(f))$ to $(H, f(\text{rad}(f))D_f)$ by Proposition 1.2(c).

Proposition 1.12 *Let $(K, A) \xrightarrow{g} (G, B) \xrightarrow{f} (H, C)$ be quadratic pair maps. Then the composite map $fg : (K, A) \rightarrow (H, C)$ is a quadratic pair map whose deviation satisfies the derivation rules (12) and (13).*

This is immediate from Proposition 1.7; it leads to the following generalization of the “quadratic envelope of the category of 2-step nilpotent groups” constructed by Jibladze and Pirashvili [23]: this is the category, denoted by **Niq**, whose objects are 2-step nilpotent groups and whose morphisms from G to H are the quadratic maps $f : G \rightarrow H$ such that $D_f \subset H'$.

Corollary 1.13 *Pairs of groups and quadratic pair maps between them form a category denoted by **QP** which we call quadratic envelope of the usual category of linear pair maps.*

In fact, the category **Niq** fully embeds into **QP** by sending G to the pair (G, G') .

Denote by **Gr** and **Ab** the category of groups and abelian groups, resp. Then the following is again an immediate consequence of Proposition 1.7.

Proposition 1.14 *Let **NQP** resp. **AQP** resp. **CP** be the full subcategory of **QP** consisting of those pairs of groups (G, B) for which B is normal in G , resp. abelian, resp. central containing G' .*

(a) *There are functors $Gr : \mathbf{QP} \rightarrow \mathbf{Ab} \times \mathbf{Gr}$, $Gr_N : \mathbf{NQP} \rightarrow \mathbf{Gr} \times \mathbf{Gr}$, $Gr_A : \mathbf{AQP} \rightarrow \mathbf{Ab} \times \mathbf{Ab}$, $Gr_C : \mathbf{CP} \rightarrow \mathbf{Ab} \times \mathbf{Ab}$ such that Gr and Gr_A send an object (G, B) to the object $(G/G'B, B)$ while Gr_N and Gr_C send (G, B) to $(G/B, B)$, and all of them send $f : (G, B) \rightarrow (H, C)$ to (\bar{f}, f_B) where $f_B : B \rightarrow C$ is the restriction of f and \bar{f} is induced by f .*

(b) *There is a bifunctor $D : \mathbf{AQP}^{op} \times \mathbf{AQP} \rightarrow \mathbf{Ab}$ defined by*

$$D((G, B), (H, C)) = \text{Hom}((G/G'B) \otimes (G/G'B), C) \quad \text{and} \quad D(g^{op}, f) = f_*(\bar{g} \otimes \bar{g})^*.$$

(c) *Assigning to a map $f : (G, B) \rightarrow (H, C)$ in **AQP** its defect $w_f \in D((G, B), (H, C))$ defines a derivation from **AQP** to D , see [2].*

Note that **CP** is contained in the intersection of **NQP** and **AQP**, and that for $(G, B) \in \mathbf{CP}$ the group G is 2-step nilpotent. Thus the category **Niq** also identifies with the full subcategory of **CP** consisting of the objects (G, G') . Moreover, **CP** is a right quadratic category in the sense of [5]; from several points of view, it plays the same role in quadratic algebra as the category of abelian groups plays in classical algebra, see [11]; it can therefore be considered as a different generalization of the category of abelian groups than the category of square groups constructed in [6]. The relation between these two generalizations, however, is not yet understood.

Universal relative quadratic map.

We now show that the functor $\text{Quad}(G, -)$ is representable; this result is also obtained in [23]. We thus get an endofunctor Q of the category of groups the properties of which are studied.

Theorem 1.15 *Let G be a group and B a subgroup of G .*

(i) *There exists a universal quadratic map relative B , $q = q_{G,B} : G \rightarrow Q(G, B)$ to some group $Q(G, B)$, i.e., for any quadratic map of groups $f : G \rightarrow H$ relative B there exists a unique linear map $\hat{f} : Q(G, B) \rightarrow H$ such that $\hat{f}q = f$.*

(ii) *The sequence of group homomorphisms*

$$0 \longrightarrow G/BG' \otimes G/BG' \xrightarrow{w_q} Q(G, B) \xrightarrow{\hat{id}} G \longrightarrow 1 \quad (14)$$

is exact. Actually, it is a central group extension represented by the bilinear 2-cocycle $D : G \times G \rightarrow G/BG' \otimes G/BG'$, $D(a, b) = -\bar{a} \otimes \bar{b}$ which corresponds to the canonical section q of \hat{id} .

Proof: The trick is to define the group $Q(G, B)$ by the cocycle D , i.e., we let $Q(G, B) = (G/BG' \otimes G/BG') \times G$ endowed with the group law $(x, a) + (y, b) = (x + y - \bar{a} \otimes \bar{b}, ab)$. Furthermore, let $q : G \rightarrow Q(G, B)$, $q(a) = (0, a)$. Then for $a, b \in G$, $q(a) + q(b) = (-\bar{a} \otimes \bar{b}, ab) = (-\bar{a} \otimes \bar{b}, 1) + q(ab)$, whence $d_q(a, b) = (\bar{a} \otimes \bar{b}, 1)$. This term is bilinear, central in $Q(G, B)$, and trivial whenever a or b is in B , so q is a quadratic map relative B with $w_q(x) = (x, 1)$. In order to prove its universal property let $f : G \rightarrow H$ be some quadratic map relative B . Define $\hat{f} : Q(G, B) \rightarrow H$ by $\hat{f}(x, a) = w_f(x) + f(a)$. Then \hat{f} satisfies $\hat{f}q = f$, and for $(x, a), (y, b) \in Q(G, B)$, we have

$$\begin{aligned} \hat{f}((x, a) + (y, b)) &= w_f(x + y - \bar{a} \otimes \bar{b}) + f(ab) \\ &= w_f(x) + w_f(y) - d_f(a, b) + d_f(a, b) + f(a) + f(b) \\ &= w_f(x) + f(a) + w_f(y) + f(b) \quad \text{since } \text{Im}(w_f) = D_f \subset Z(I_f) \\ &= \hat{f}(x, a) + \hat{f}(y, b). \end{aligned}$$

To prove uniqueness of \hat{f} let $g : Q(G, B) \rightarrow H$ be a linear map such that $gq = f$. Then $g(0, a) = gq(a) = f(a)$ and $g(\bar{a} \otimes \bar{b}, 1) = g d_q(a, b) = d_{gq}(a, b)$ (by 1.8) $= d_f(a, b) = w_f(\bar{a} \otimes \bar{b})$, whence by linearity, $g(x, 1) = w_f(x)$ for all $x \in G/BG' \otimes G/BG'$. Thus $g(x, a) = g((x, 1) + (0, a)) = w_f(x) + f(a) = \hat{f}(x, a)$, whence $g = \hat{f}$. Finally, we have $\hat{id}(x, a) = w_{id}(x)a = a$ as id is linear, so \hat{id} is the projection to the second factor which proves exactness of the sequence in (ii) as we saw already that w_q is the canonical injection of the first factor. \square

Write $Q(G) = Q(G, \{1\}) = Q(G, G')$ and $q = q_G = q_{G, \{1\}}$. Note that Theorem 1.15(i) says that the map

$$q^* : \text{Hom}(Q(G), H) \rightarrow \text{Quad}(G, H) \quad (15)$$

is a bijection natural in H .

Corollary 1.16 *Let X be a set and F a free group with basis X . Then there is a (non natural) isomorphism $\phi : Q(F) \rightarrow (F^{ab} \otimes F^{ab}) \times F$ such that ϕq_F is given by*

$$\phi q_F \left(\prod_{i=1}^n x_i^{\epsilon_i} \right) = \left(\sum_{i=1}^n \frac{1 - \epsilon_i}{2} \bar{x}_i \otimes \bar{x}_i + \sum_{1 \leq i < j \leq n} \epsilon_i \epsilon_j \bar{x}_i \otimes \bar{x}_j, \prod_{i=1}^n x_i^{\epsilon_i} \right) \quad (16)$$

for $x_i \in X$, $\epsilon_i = \pm 1$. Consequently, for any group H there is a bijection

$$Quad(F, H) \cong \{(\chi, \psi) \in H^X \times H^{X \times X} \mid [\text{Im}(\chi), \text{Im}(\chi)] = [\text{Im}(\chi), \text{Im}(\psi)] = \{1\}\} \quad (17)$$

which carries $f \in Quad(F, H)$ to $(f|_X, d_{f|_{X \times X}})$.

Proof: The map $q_{F|_X}$ induces a splitting of the central extension (14). The formula for ϕq_F is obtained using (2) and (7). To determine $Quad(F, H)$ now use the bijection q_F^* in (15) and the fact that $X \times X \rightarrow F^{ab} \otimes F^{ab}$, $(x, y) \mapsto x \otimes y$, is a basis of the abelian group $F^{ab} \otimes F^{ab}$. \square

Proposition 1.17 *There is an endofunctor Q of the category of groups sending G to $Q(G)$ and $f : G \rightarrow H$ to $Q(f) = \widehat{q_H f} : Q(G) \rightarrow Q(H)$. It has the following properties.*

(a) *If G is n -step nilpotent for $n \geq 2$ then so is $Q(G)$.*

(b) *If $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \rightarrow 1$ is an exact sequence of groups then so is the sequence*

$$Q(G_1) \times G_1^{ab} \otimes G_2^{ab} \times G_2^{ab} \otimes G_1^{ab} \xrightarrow{\xi} Q(G_2) \xrightarrow{Q(\beta)} Q(G_3) \rightarrow 1 \quad (18)$$

where $\xi(x, y, z) = Q(\alpha)(x) + w_q(\alpha^{ab} \otimes 1)(y) + w_q(1 \otimes \alpha^{ab})(z)$. One also has the identity

$$\text{Ker}(Q(\beta)) = \text{Im}(q_{G_2} \alpha) + \text{Im}(w_{q_{G_2}}(\alpha \otimes \alpha + \alpha^{ab} \otimes 1_{G_2} + 1_{G_2} \otimes \alpha^{ab})) \quad (19)$$

Similarly, assigning the group $Q(G, B)$ to a pair (G, B) defines a functor from the category of linear pair maps to the category of groups.

Proof: (a): Let $a \in Q(G)$, $b \in \gamma_n(Q(G))$; we must show that $[a, b] = 1$. Modulo the central subgroup $\text{Im}(w_q)$ we may assume that $a = qa'$, $b = qb'$ for some $a', b' \in G$. Then by (9), $[a, b] = q[a', b'] - w_q(\bar{a}' \otimes \bar{b}' - \bar{b}' \otimes \bar{a}') = 1$ since $[a', b'] = 1$ as $b' \in \gamma_n(G)$ and since $\bar{b}' = 0$ as $n \geq 2$.

For (b) use naturality of sequence (14) with respect to the maps α and β ; then exactness of sequence (18) follows by an easy diagram chase together with right exactness of the tensor product. Identity (19) then follows using the fact that $Q(G_1) = \langle \text{Im}(q) \rangle + \text{Im}(w_q)$. \square

Corollary 1.18 *If G is generated by a subset X then $Q(G)$ is generated by the subset $q(X) \cup d_q(X \times X)$.*

Just apply Proposition 1.17(b) to the epimorphism from the free group of basis X to G and use Corollary 1.16.

Corollary 1.19 *Two quadratic maps $f, g : G \rightarrow H$ coincide if and only if they coincide on a generating subset X of G and d_f coincides with d_g on $X \times X$.*

Just note that $f = g$ iff $\hat{f} = \hat{g}$ as q is injective (having \hat{id} as a retraction).

The above properties of the functor Q allow to construct quadratic maps on groups defined by generators and relations, as follows. Let $G = \langle X | R \rangle$ be a presentation of G , i.e. let X be a set, F the free group with basis X , R a subset of F and $\pi : F \twoheadrightarrow G$ a homomorphism whose kernel is the normal subgroup of F generated by R . For $r \in R$, write $r = \prod_{i=1}^{n_r} x_{ri}^{\epsilon_{ri}}$ with $x_{ri} \in X$ and $\epsilon_{ri} = \pm 1$, and $\bar{r} = rF' = \sum_{x \in X} k_{rx} \bar{x}$ in F^{ab} , with $k_{rx} \in \mathbb{Z}$.

Proposition 1.20 *For some group H let $(\chi, \psi) \in H^X \times H^{X \times X}$. Then there exists a quadratic map $f : G \rightarrow H$ such that*

$$f\pi(x) = \chi(x) \quad \text{and} \quad d_f(\pi(x), \pi(y)) = \psi(x, y) \quad \text{for } x, y \in X \quad (20)$$

if and only if for all $r \in R$ and $y \in X$ the following three conditions hold.

- (i) $[\text{Im}(\chi), \text{Im}(\chi)] = [\text{Im}(\chi), \text{Im}(\psi)] = \{1\}$
- (ii) $\sum_{i=1}^{n_r} \left(\epsilon_{ri} \chi(x_{ri}) + \frac{1-\epsilon_i}{2} \psi(x_{ri}, x_{ri}) \right) + \sum_{1 \leq i < j \leq n} \epsilon_{ri} \epsilon_{rj} \psi(x_{ri}, x_{rj}) = 0$
- (iii) $\sum_{x \in X} k_{rx} \psi(x, y) = \sum_{x \in X} k_{rx} \psi(y, x) = 0$

Moreover, any quadratic map from G to H is induced by maps χ and ψ as above.

Proof: Let F_1 be the free group with basis $F \times R$. Then the sequence $F_1 \xrightarrow{\delta} F \xrightarrow{\pi} G \rightarrow 1$ is exact with $\delta(a, r) = {}^a r$ for $(a, r) \in F \times R$. Hence by (15) and Proposition 1.17(b) there exists a quadratic map $f : G \rightarrow H$ satisfying (20) iff there is a homomorphism $\kappa : Q(F) \rightarrow H$ factoring through $Q(\pi)$ and satisfying the property

$$\kappa q_F(x) = \chi(x) \quad \text{and} \quad \kappa d_{q_F}(x, y) = \psi(x, y) \quad \text{for } x, y \in X. \quad (21)$$

By Corollary 1.16 a homomorphism κ satisfying (21) exists iff condition (i) holds, so suppose this true in the sequel. By Proposition 1.17(b) κ factors through $Q(\pi)$ iff $\kappa Q(\delta) = \kappa w_{q_F}(\delta^{ab} \otimes 1_{F^{ab}} + 1_{F^{ab}} \otimes \delta) = 0$. But $\text{Im}(\delta^{ab}) = \langle \text{Im}(R \rightarrow F^{ab}) \rangle$ whence

the second identity holds iff for all $(r, x) \in R \times X$, $\kappa w_{q_F}(\bar{r}, \bar{x}) = \kappa w_{q_F}(\bar{x}, \bar{r}) = 0$ which by expanding \bar{r} is equivalent to condition (iii). Next consider $\kappa Q(\delta)$. By Corollary 1.18 $Q(F_1)$ is generated by $q_F \delta(F \times R) + \text{Im}(w_{q_F}(\delta^{ab} \otimes \delta^{ab}))$. Let $(a, r) \in F \times R$. Then $q_F \delta(a, r) = q_F({}^a r) = {}^{q_F(a)} q_F(r) + w_{q_F}(\bar{a} \otimes \bar{r} - \bar{r} \otimes \bar{a})$ by (10). Hence if κ satisfies conditions (iii) it annihilates $\text{Im}(Q(\delta))$ iff it annihilates $q_F(R)$ which is equivalent to condition (ii) by (16). \square

2 Relation with Passi's construction.

In this section we study relative polynomial maps in the sense of Passi by using the nonabelian theory of the first section as an essential tool. We will see that the proof of the main properties in the quadratic case becomes more natural in this way than in former approaches in the literature.

For a commutative ring R with unit let $I_R(G)$ denote the augmentation ideal of the group algebra $R(G)$; for $k \geq 0$, $I_R^k(G)$ denotes its k -th power, with the convention $I_R^0(G) = R(G)$. If $R = \mathbb{Z}$ we also write $I(G) = I_{\mathbb{Z}}(G)$. For a function $f : G \rightarrow A$ to some *abelian* group A let $\bar{f} : \mathbb{Z}(G) \rightarrow A$ denote the extension of f to a \mathbb{Z} -linear homomorphism.

Definition 2.1 *Let G be a group and B be a normal subgroup of G . We say that a function $f : G \rightarrow A$ as above is polynomial of degree $\leq n$ relative B if \bar{f} annihilates the subset $1 + I(B)I(G) + I^{n+1}(G)$ of $\mathbb{Z}(G)$.*

Moreover, we say that f is (normalized) polynomial of degree $\leq n$ if it is polynomial of degree $\leq n$ relative $\{1\}$.

Remark 2.2 The notion of (absolute) polynomial map from groups to *abelian* groups is due to Passi [30]. The relative case was only implicitly present in most of the work in the literature based on this notion, notably when related to the dimension subgroup problem. See [32] for a thorough treatment of the subject.

Note that f is polynomial of degree ≤ 0 iff $f = 0$. Moreover, if f is polynomial of degree $\leq n$, f is also polynomial of degree $\leq n$ relative $\gamma_n(G)$ as $I(\gamma_n(G)) \subset I^n(G)$; this is immediate by inductive application of the formula

$$[a, b] - 1 = [a - 1, b - 1]a^{-1}b^{-1} \quad (22)$$

for $a, b \in G$ where $[-, -]$ denotes the group commutator on the left and the ring commutator on the right.

The following inductive characterization of polynomial maps is useful for proving polynomiality, see [14] where a more general version is developed (with respect to arbitrary N -series). For convenience of the reader we give a direct proof of our special case here which is very short and easy anyway.

Proposition 2.3 *Let $f : G \rightarrow A$ be any normalized function from a group G to some abelian group A . Then we have the following properties.*

- (1) *For $a, b \in G$, $d_f(a, b) = \bar{f}((a-1)(b-1))$.*
- (2) *For $n \geq 1$, f is polynomial of degree $\leq n$ relative some given normal subgroup B of G if and only if the following two conditions hold.*
 - (a) *The map $d_f(a, -) : G \rightarrow A$ (or equivalently, $d_f(-, a) : G \rightarrow A$) is polynomial of degree $\leq n-1$ for all $a \in G$.*
 - (b) *For all $(b, a) \in B \times G$, $f(ba) = f(b) + f(a)$ or equivalently, $d_f(b, a) = 0$.*

Proof: (1) is immediate from expanding the right hand term. To prove (2) let $a \in G$. Then for $b \in G$, $\overline{d_f(a, -)}(b-1) = d_f(a, b) - d_f(a, 1) = \bar{f}((a-1)(b-1))$ by (1). As the elements $b-1, b \in G$, generate $I(G)$ as a \mathbb{Z} -module, it follows by linearity that $\overline{d_f(a, -)}(x) = \bar{f}((a-1)x)$ for all $x \in I(G)$. Hence $\overline{d_f(a, -)}(I^n(G)) = \bar{f}((a-1)I^n(G))$, which implies that property (a) is equivalent to f being polynomial of degree $\leq n$. Moreover, (b) is equivalent to $\bar{f}((b-1)(a-1)) = d_f(b, a) = 0$ for all $(b, a) \in B \times G$ which in turn means that $\bar{f}(I(B)I(G)) = 0$. \square

In low degrees, we obtain the following characterization of (relative) polynomial maps.

Corollary 2.4 *Let $f : G \rightarrow A$ and B as in 2.3.*

- (1) *f is polynomial of degree 1 relative B iff it is linear.*
- (2) *f is polynomial of degree 2 relative B iff d_f is bilinear and annihilates B in the first variable.* \square

Corollary 2.5 *Let G be a group, B be a central subgroup of G and $f : G \rightarrow A$ as in 2.3. Then f is polynomial of degree ≤ 2 relative B if and only if f is quadratic relative B .*

This is immediate from 2.4, just note that centrality of B implies that $d_f(a, b) = \bar{f}((a-1)(b-1)) = \bar{f}((b-1)(a-1)) = d_f(b, a)$ for $(a, b) \in G \times B$. \square

Before exploiting corollary 2.5 we give some examples the verification of which is based on 2.3 and 2.4.

Examples 2.6 Let G be a group.

- (1) If G is 3-step nilpotent, the commutator map $G \times G \rightarrow G$, $(a, b) \mapsto [a, b]$, is bipolynomial of degree ≤ 2 .
- (2) Recall that the non-abelian tensor square $G \otimes G$ of G is a group closely related to the homotopy group $\pi_3 \Sigma K(G, 1)$ and also to the second homology group $H_2(G)$,

see [8] and [9]. Now if G is 2-step nilpotent, the natural map $G \times G \rightarrow G \otimes G$, $(a, b) \mapsto a \otimes b$, is bipolynomial of degree ≤ 2 , see [16] where this fact is used to compute $G \otimes G$ for 2-step nilpotent groups.

(3) For G abelian the n -fold diagonal map $\delta^n : G \rightarrow G^{\otimes n}$, $\delta^n(a) = a \otimes \cdots \otimes a$, is polynomial of degree $\leq n$, see [14].

In order to introduce *universal* relative polynomial maps we recall the following definition and facts from [15].

Definition 2.7 Let G be a group, R as above, and $B \triangleleft G$ a normal subgroup. We define the quotient R -algebra without unit

$$P_{n,R}(G, B) = I_R(G) / (I_R(B)I_R(G) + I_R^{n+1}(G)) .$$

The canonical quotient map from $I_R(G)$ to $P_{n,R}(G)$ or to $P_{n,R}(G, B)$ is denoted by ρ or ρ_n . If $f : (G, B) \rightarrow (H, C)$ is a linear pair map in **NQP** it induces a morphism of R -algebras $P_{n,R}(f) : P_{n,R}(G, B) \rightarrow P_{n,R}(H, C)$ defined by $P_{n,R}(f)\rho(a - 1) = \rho(f(a) - 1)$ for $a \in G$.

We point out that $P_{n,R}(G) = P_{n,R}(G, \{1\})$ is the polynomial group constructed by Passi [30]; the relative version was introduced and studied modulo torsion in [15]. The aim of this section is to explore in detail the structure of $P_{2,\mathbb{Z}}(G, B)$ for central B which is the crucial ingredient of our abelian models for 2-step nilpotent groups in [13], [17].

Using the elementary identification

$$I_R(G)/I_R(B)I_R(G) \xrightarrow{\cong} I_R(G/B), \quad \overline{a-1} \mapsto \bar{a} - 1 \quad \text{for } a \in G, \quad (23)$$

we see that multiplication in the ring $P_{n,R}(G, B)$ gives rise to an R -linear map

$$\mu_n : P_{n-1,R}(G/B) \otimes_{R(G)} P_{n-1,R}(G, B) \longrightarrow P_{n,R}(G, B) \quad (24)$$

such that for $x, y \in I_R(G)$, $\mu_n(\rho_{n-1}(x) \otimes \rho_{n-1}(y)) = \rho_n(xy)$. This shows that via left multiplication, $P_{n,R}(G, B)$ is a nilpotent left $R(G/B)$ -module of class $\leq n$; recall that a left $R(G)$ -module B is called *nilpotent of class $\leq k$* if $I_R^k(G) \cdot B = 0$.

Now consider the map

$$p_{n,R} : G \rightarrow P_{n,R}(G, B), \quad p_{n,R}(a) = \rho(a - 1) .$$

In the case where $R = \mathbb{Z}$ we omit the subscript R . Recall that for a group homomorphism $f : G \rightarrow H$ and an $R(H)$ -module M , an *f -derivation* from G to M is a map $d : G \rightarrow M$ such that $d(ab) = f(a)d(b) + d(a)$ for $a, b \in G$.

Proposition 2.8 *The maps $p_{n,R}$ and p_n have the following universal properties:*

(i) $p_{n,R} : G \rightarrow P_{n,R}(G, B)$ is a universal $(G \twoheadrightarrow G/B)$ -derivation from G into nilpotent $R(G/B)$ -modules of class $\leq n$.

(ii) $p_n : G \rightarrow P_n(G, B)$ is a universal polynomial map of degree $\leq n$ relative B from G into abelian groups.

Proof: (i) follows from the well known fact that the map $G \rightarrow I_{n,R}(G)$, $a \mapsto a-1$, is a universal derivation from G into arbitrary G -modules, see [22] VI.5. To prove (ii) we first show that p_n is polynomial of degree $\leq n$ relative B . The linear extension $\bar{p}_n : \mathbb{Z}(G) \rightarrow P_n(G, B)$ of p_n satisfies $\bar{p}_n(a-1) = \rho(a-1)$ for $a \in G$, so $\bar{p}_n(x) = \rho(x)$ for all $x \in I(G)$ by linearity. In particular, $\bar{p}_n(1 + I(B)I(G) + I^{n+1}(G)) = \rho(I(B)I(G) + I^{n+1}(G)) = 0$, whence the assertion. To prove the universal property, let $f : G \rightarrow A$ be any polynomial map of degree $\leq n$ relative B . Then $\bar{f} : \mathbb{Z}(G) \rightarrow A$ factors through a \mathbb{Z} -linear map, also denoted by \bar{f} , $\mathbb{Z}(G)/(I(B)I(G) + I^{n+1}(G)) \rightarrow A$. Denoting the restriction of \bar{f} to $P_n(G, B)$ again by \bar{f} we have $\bar{f}p_n = f$ as $\bar{f}(1) = 0$. Furthermore, \bar{f} is the unique \mathbb{Z} -linear map with this property since the elements $p_n(a)$, $a \in G$, generate $P_n(G, B)$ as a \mathbb{Z} -module. \square

Property (i) implies a canonical isomorphism $P_{n,R}(G, B) \cong R(G/B) \otimes_{\mathbb{Z}(G)} P_n(G)$ of left $R(G/B)$ -modules.

Let $f : G \rightarrow A$ be a polynomial map of degree $\leq n$ relative B and $\bar{f} : P_n(G, B) \rightarrow A$ the canonical induced \mathbb{Z} -linear map according to property (ii). Define the homomorphism

$$w_f = \bar{f}\mu_n : P_{n-1}(G/B) \otimes_{\mathbb{Z}(G)} P_{n-1}(G, B) \longrightarrow A. \quad (25)$$

Then by 2.3 (1) we have the following commutative diagram of factorizations induced by f .

$$\begin{array}{ccccc} G & \xrightarrow{f} & A & \xleftarrow{d_f} & G \times G \\ \parallel & & \uparrow \bar{f} & \nwarrow w_f & \downarrow p_{n-1} \times p_{n-1} \\ G & \xrightarrow{p_n} & P_n(G, B) & \xleftarrow{\mu_n} & P_{n-1}(G/B) \otimes_{\mathbb{Z}(G)} P_{n-1}(G, B) \end{array} \quad (26)$$

Note that $w_{p_n} = \mu_n$.

Finally, suitable group homomorphisms induce ring homomorphisms on the constructions introduced above in the obvious way.

From (23) and the inclusion $I_R(\gamma_n(G)) \subset I_R^n(G)$ we deduce the natural isomorphisms of R -algebras

$$P_{n,R}\left(G/\gamma_{n+1}(G), B\gamma_{n+1}(G)/\gamma_{n+1}(G)\right) \xleftarrow{\cong} P_{n,R}(G, B) \xrightarrow{\cong} P_{n,R}(G, B\gamma_n(G)) \quad (27)$$

Now let

$$\underline{G}: B \xrightarrow{i} G \xrightarrow{\pi} Q \quad (28)$$

be a group extension with abelian kernel B . We will frequently identify B with $i(B)$ and suppress i from the notation. It is an elementary fact that the sequence

$$R \otimes_{\mathbb{Z}} B \xrightarrow{p_{n,R}i} P_{n,R}(G, B) \xrightarrow{P_{n,R}(\pi)} P_{n,R}(Q) \rightarrow 0 \quad (29)$$

is an exact sequence of $R(Q)$ -linear homomorphisms, where the Q -action on $R \otimes_{\mathbb{Z}} B$ is given by R -linear extension of the Q -action on B induced by conjugation in G . Moreover, the map $p_{n,R}i$ here denotes the R -linear extension of the map $p_{n,R}i$ defined above.

Now consider the case where B is *central* in G . Then $I_R(B)I_R(G) = I_R(G)I_R(B)$, whence the map μ_n in (24) factors through another R -linear map, also denoted by μ_n ,

$$\mu_n : P_{n-1,R}(G/B) \otimes_{R(G/B)} P_{n-1,R}(G/B) \rightarrow P_{n,R}(G, B) \quad (30)$$

such that $\mu_n(p_{n-1,R}(a) \otimes p_{n-1,R}(b)) = p_{n,R}(a)p_{n,R}(b)$, $a, b \in G$. Consequently, if $f : G \rightarrow A$ is a polynomial map of degree $\leq n$ relative B then w_f in (25) factors through another \mathbb{Z} -linear map, also denoted by w_f ,

$$w_f : P_{n-1}(G/B) \otimes_{\mathbb{Z}(G/B)} P_{n-1}(G/B) \rightarrow A \quad (31)$$

such that $w_f(p_{n-1}(\bar{a}) \otimes p_{n-1}(\bar{b})) = d_f(a, b)$, $a, b \in G$.

Now consider the case we are mainly interested in here, that is $n = 2$ and B central in G . Noting that $P_1(G/B)$ is a trivial G/B -module and using the canonical identifications $P_1(G/B) \cong (G/B)^{ab} \cong G/BG'$ we see that here w_f is equivalent to the following linear map, also denoted by w_f ,

$$w_f : G/BG' \otimes G/BG' \xrightarrow{\cong} P_1(G/B) \otimes_{\mathbb{Z}(G/B)} P_1(G/B) \rightarrow A. \quad (32)$$

This map satisfies $w_f(\bar{a} \otimes \bar{b}) = d_f(a, b)$ for $a, b \in G$, whence it coincides with the map w_f defined in Proposition 1.2(d) so that there is essentially no ambiguity in our notation.

Now we are ready for comparing the two universal constructions for quadratic and degree 2 polynomial maps, respectively.

Proposition 2.9 *Let B be a central subgroup of a group G . Then there is a natural isomorphism of abelian groups $\alpha : Q(G, B)^{ab} \xrightarrow{\cong} P_2(G, B)$ such that $\alpha \circ ab \circ q = p_2$ and $\alpha \circ ab \circ w_q = \mu_2$.*

Proof : By 2.5 the map $p_2 : G \rightarrow P_2(G, B)$ is quadratic relative B and the map $ab \circ q : G \rightarrow Q(G, B)^{ab}$ is polynomial of degree ≤ 2 . Hence α and its inverse are induced by the universal properties of q and p_2 , resp. So the equation $\alpha \circ ab \circ q = p_2$

holds by construction. Then the second one follows from the definition of w_q and from the identity $w_{p_n} = \mu_n$. \square

Let G be a group and B a central subgroup of G . Then we have natural homomorphisms of abelian groups

$$BG'/\gamma_3(G) \xleftarrow{c_2} G/BG' \wedge G/BG' \xrightarrow{l_2} G/BG' \otimes G/BG' \quad (33)$$

defined by $c_2(\bar{a} \wedge \bar{b}) = [a, b]\gamma_3(G)$ and $l_2(\bar{a} \wedge \bar{b}) = \bar{a} \otimes \bar{b} - \bar{b} \otimes \bar{a}$.

Theorem 2.10 *Let G be a group and B a central subgroup of G . Then the following natural sequences of abelian groups are exact.*

$$0 \longrightarrow \text{Ker}(c_2) \xrightarrow{l_2} G/BG' \otimes G/BG' \xrightarrow{\mu_2} P_2(G, B) \xrightarrow{\rho_1} G^{ab} \longrightarrow 1 \quad (34)$$

$$0 \rightarrow G/BG' \wedge G/BG' \xrightarrow{(c_2, -l_2)^t} BG'/\gamma_3(G) \oplus (G/BG' \otimes G/BG') \xrightarrow{(p_2^i, \mu_2)} P_2(G, B) \xrightarrow{\rho_2} G/BG' \rightarrow 1 \quad (35)$$

$$0 \longrightarrow B\gamma_3(G)/\gamma_3(G) \xrightarrow{p_2^i} P_2(G, B) \xrightarrow{P_2(\pi)} P_2(G/B) \longrightarrow 0 \quad (36)$$

where $\rho_1 p_2(a) = aG'$ and $\rho_2 p_2(a) = aBG'$ for $a \in G$, $j: I^2(G)/I^3(G) \hookrightarrow I(G)/I^3(G) = P_2(G)$, i denotes the injection of $BG'/\gamma_3(G)$ or $B\gamma_3(G)/\gamma_3(G)$ into $G/\gamma_3(G)$, and $\pi: (G, B) \rightarrow (G/B, \{\bar{1}\})$ is the natural projection.

Remarks 2.11 Taking $B = \{1\}$ in (34) we rediscover the natural isomorphism $I^2(G)/I^3(G) \cong \text{U}_2\text{L}(G)$ in [1]. Moreover, centrality of B is a crucial hypothesis in Theorem 2.10 as in general (36) has to be replaced by the natural exact sequence

$$\text{Tor}_1^{\mathbb{Z}}(G/BG', G/BG') \xrightarrow{[\cdot, \cdot]\tau} B\gamma_3(G)/B'\gamma_3(G) \xrightarrow{p_2^i} P_2(G, B) \xrightarrow{P_2(\pi)} P_2(G/B) \longrightarrow 0$$

where the map $[\cdot, \cdot]\tau$ sends a typical generator $\langle aBG', k, bBG' \rangle$ with $a, b \in G$, $k \in \mathbb{Z}$ such that $a^k, b^k \in BG'$ (see [29, V.6]), to the element $[a, b^k]B'\gamma_3(G)$ which is nontrivial in general, see Theorem 2.6 and Example 2.4 in [18].

Proof of Theorem 2.10: By the right hand isomorphism in (27) we may assume that $\gamma_3(G) = 1$. Now recall the central extension

$$0 \longrightarrow G/BG' \otimes G/BG' \xrightarrow{w_q} Q(G, B) \xrightarrow{\hat{id}} G \longrightarrow 1$$

from 1.15. Putting $P = \hat{id}^{-1} G'$ and writing $[-, -]$ for the respective commutator maps we have the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
Q(G, B) \times Q(G, B) & \xrightarrow{\widehat{id} \times \widehat{id}} & G \times G \\
\downarrow [-, -] & \swarrow c & \downarrow [-, -] \\
G/BG' \otimes G/BG' & \xrightarrow{w_q} & P & \xrightarrow{\widehat{id}} & G' \\
\parallel & & \downarrow inc & & \downarrow inc \\
G/BG' \otimes G/BG' & \xrightarrow{w_q} & Q(G, B) & \xrightarrow{\widehat{id}} & G
\end{array}$$

The factorisation through c of the commutator map of $Q(G, B)$ exists as $\text{Im}(w_q)$ is central in $Q(G, B)$; it satisfies

$$c(a, b) = [q(a), q(b)] = q[a, b] - w_q(\bar{a} \otimes \bar{b} - \bar{b} \otimes \bar{a}) \quad (37)$$

by (9). As $[-, -] : G \times G \rightarrow G'$ is bilinear and q is linear on G' by Proposition 1.2(c), c is bilinear; it annihilates BG' in both variables since B is central and G' is abelian. Now P is abelian being a split central extension of the abelian group G' by construction of $Q(G, B)$. Hence c gives rise to a homomorphism $\bar{c} : G/BG' \wedge G/BG' \rightarrow P$ such that $\bar{c}(\bar{a} \wedge \bar{b}) = [q(a), q(b)]$ and $Q(G, B)' = \langle \text{Im}(c) \rangle = \text{Im}(\bar{c})$. Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
G/BG' \otimes G/BG' & \xrightarrow{w_q} & P & \xrightarrow{\widehat{id}} & G' \\
\uparrow -l_2 & & \uparrow \bar{c} & & \parallel \\
\text{Ker}(c_2) & \xrightarrow{inc} & G/BG' \wedge G/BG' & \xrightarrow{c_2} & G'
\end{array}$$

The right hand square clearly commutes, and by (37), $\bar{c} = qc_2 - w_q l_2$, whence the left hand square commutes, too. It follows that $Q(G, B)^{ab} = \text{coker}(inc \circ \bar{c})$ fits into the exact sequence

$$0 \rightarrow (G/BG' \otimes G/BG') / l_2 \text{Ker}(c_2) \xrightarrow{ab \circ w_q} Q(G, B)^{ab} \rightarrow G'^{ab} \rightarrow 1$$

which becomes sequence (34) under the identification $Q(G, B)^{ab} \cong P_2(G, B)$, see 2.9. As to sequence (35), first note that it is exact in $P_2(G, B)$ since the composite isomorphism $\text{coker}(\mu_2) \cong I(G)/I^2(G) \cong G/G'$ takes $p_2 i(b)$ to bG' for $b \in B$. Furthermore, we have $(p_2 i, \mu_2)(c_2, -l_2)^t = 0$ by the relation $p_2 i c_2 = \mu_2 l_2$ which follows from the identity

$$p_2([a, b]) = [p_2(a), p_2(b)] = p_2(a)p_2(b) - p_2(b)p_2(a) \quad (38)$$

for $a, b \in G$; this is immediate from equation (22). The map $(c_2, -l_2)^t$ is injective as l_2 is; it remains to show that the map $\overline{(p_2 i, \mu_2)}: \Pi := \text{coker}(c_2, -l_2)^t \rightarrow P_2(G, B)$ induced by $(p_2 i, \mu_2)$ is injective. Consider the commutative diagram (39) below where ϕ is the natural projection and i_1, i_2 are the natural inclusions into $BG' \oplus (G/BG' \otimes G/BG')$ followed by the natural projection to Π .

$$\begin{array}{ccccc}
G/BG' \wedge G/BG' & \xrightarrow{c_2} & BG' & \xrightarrow{\phi} & BG'/G' \\
\downarrow l_2 & & \downarrow i_1 & & \parallel \\
G/BG' \otimes G/BG' & \xrightarrow{i_2} & \Pi & \xrightarrow{\overline{(\phi, 0)}} & BG'/G' \\
\downarrow & & \parallel & & \parallel \\
G/BG' \otimes G/BG' / l_2 \text{Ker}(c_2) & \xrightarrow{\overline{i_2}} & \Pi & \xrightarrow{\overline{(\phi, 0)}} & BG'/G' \\
\parallel & & \downarrow \overline{(p_2 i, \mu_2)} & & \downarrow inc \\
G/BG' \otimes G/BG' / l_2 \text{Ker}(c_2) & \xrightarrow{\overline{\mu_2}} & P_2(G, B) & \xrightarrow{\rho_1} & G^{ab}
\end{array} \tag{39}$$

The rows are exact: for the bottom row this follows from sequence (34), for the second and third row from the fact that the upper left hand square is a cocartesian square of abelian groups (or by easy direct arguments). Injectivity of $\overline{(p_2 i, \mu_2)}$ is now immediate.

Finally, to prove exactness of sequence (36) it suffices to check injectivity of $p_2 i$, see (29). But in the above diagram i_1 is injective as l_2 is, so $p_2 i = \overline{(p_2 i, \mu_2)} \circ i_1$ is injective, too. \square

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